Interactive Proofs

Or how I stopped worrying and learned to ask questions

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- Here, the verifier V, is a polynomial time Turing machine which takes strings of a language L and outputs 1 if the string is in L or 0 otherwise.

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- Here, the verifier V, is a polynomial time Turing machine which takes strings of a language L and outputs 1 if the string is in L or 0 otherwise.

- The Prover P, is a <u>function</u> that maps strings to a certificate or "Sorry, not in the language".

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- Provers are always trying to prove correctness, even if a statement is not correct.

- Even if the prover diligently says that there is no proof, the verifier cannot be sure unless the verifier knows that the prover is **all powerful**.

Interactive Proof systems: The Protocol

Definition: Let $f, g : \{0, 1\}^* \rightarrow \{0, 1\}^*$ be functions. A k-round interaction of f and g on input $x \in \{0, 1\}^*$, denoted by $\langle f, g\rangle(x)$ is the sequence of the following strings $a_1, \ldots, a_k \in \{0, 1\}^*$ defined as follows:

$$a_1 = f(x)$$

$$a_2 = g(x, a_1)$$

$$\ldots$$

$$a_{2i+1} = f(x, a_1, \ldots, a_{2i})$$

$$a_{2i+2} = g(x, a_1, \ldots, a_{2i+1})$$

The output of f at the end of the interaction, $out_f\langle f, g\rangle(x)$, is defined to be

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Transcript

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Interactive Proof systems

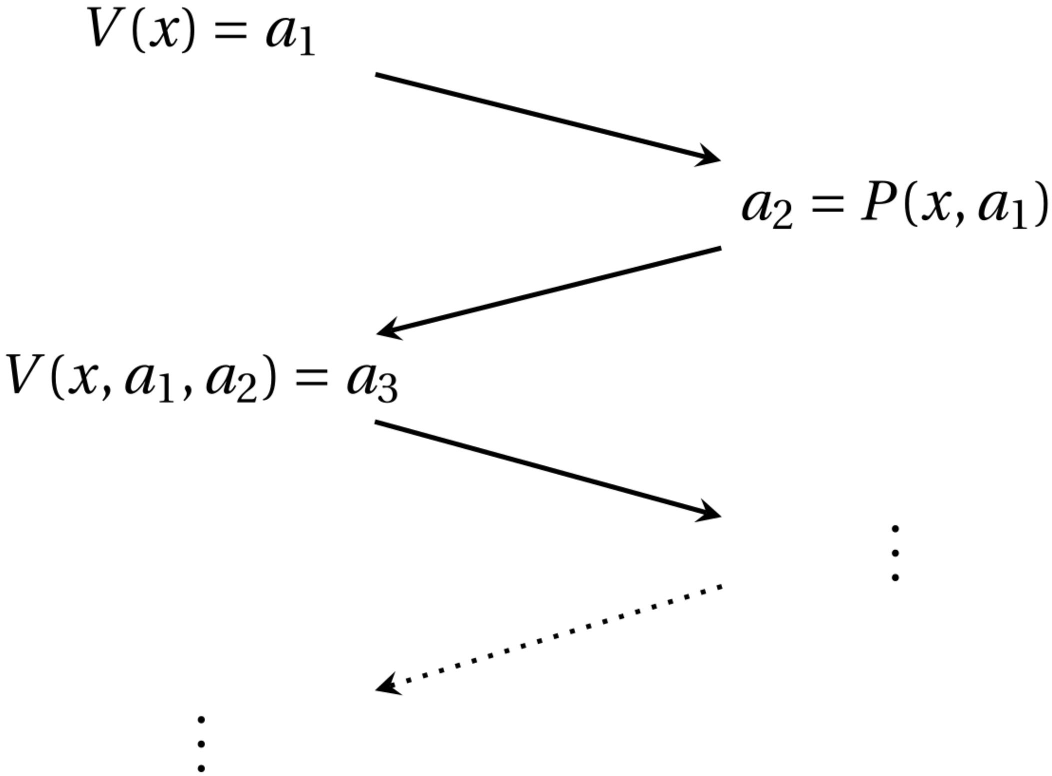
Definition: Deterministic proof systems

For $k \geq 1$, We say that a language L has a k-round deterministic interactive proof system if there's a deterministic poly-time TM V that on input $x, a_1, \ldots, a_i$ runs in time polynomial in $|x|$, satisfying:

$$x \in L \Rightarrow \exists P : \{0,1\}^* \rightarrow \{0,1\}^* \ out_V \langle V, P\rangle(x) = 1 \text{ (Completeness)}$$

$$x \notin L \Rightarrow \forall P : \{0,1\}^* \rightarrow \{0,1\}^* \ out_V \langle V, P\rangle(x) = 0 \text{ (Soundness)}$$

Interactive Proof systems



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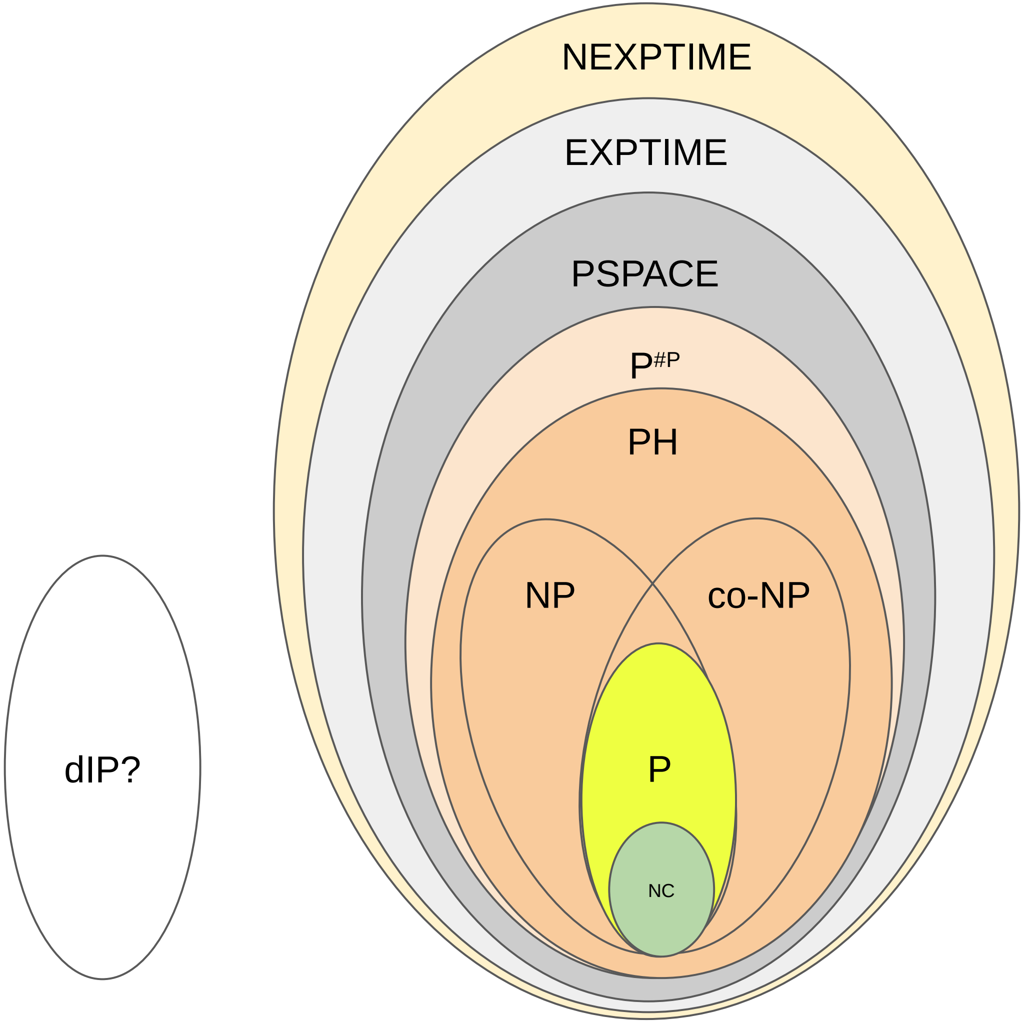
- **dIP** is the set of all languages with poly(n)-round deterministic interactive proof system.

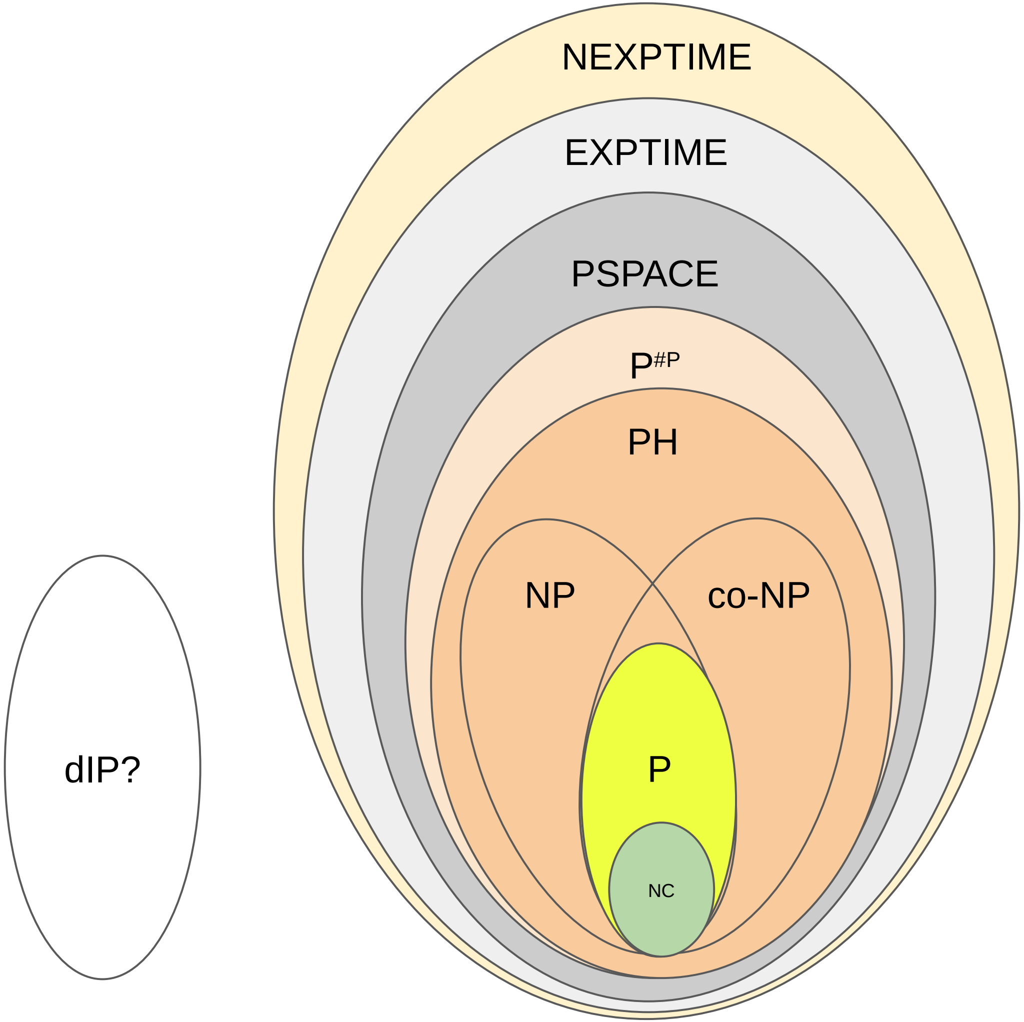
- Can't we define a class of constant round deterministic interactive proof systems?

Where is dIP?



Where is dIP?

- Claim: **NP $\subseteq$ dIP**



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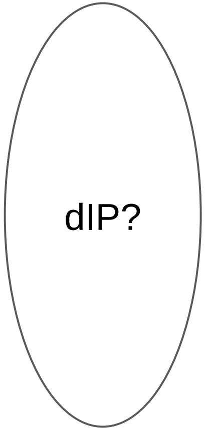
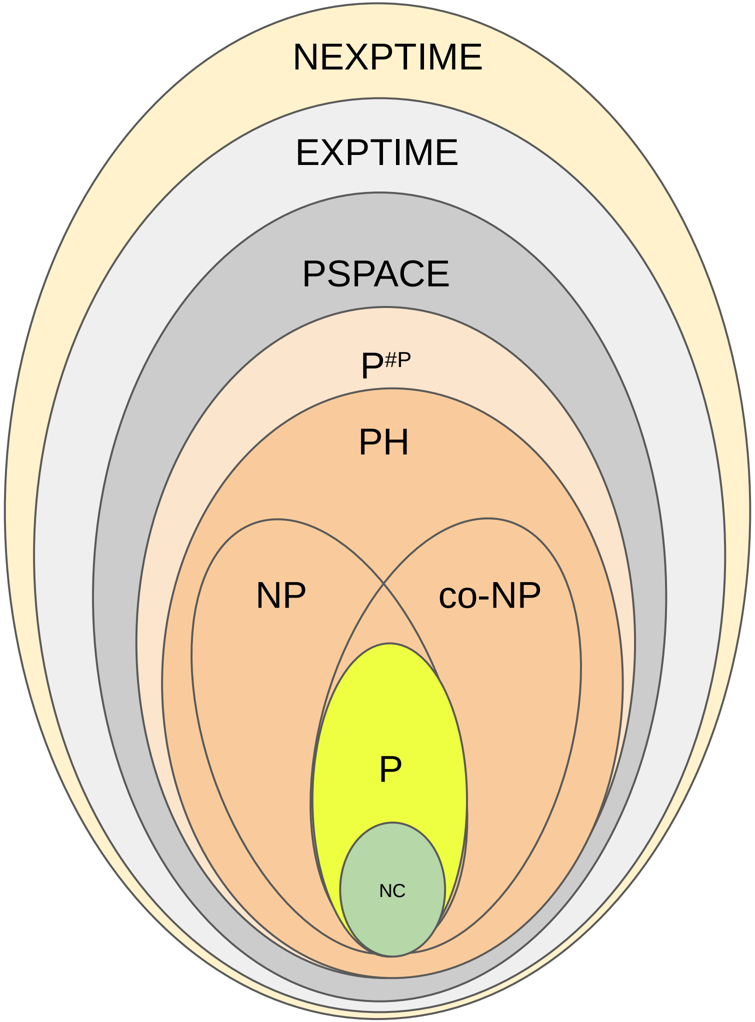
Proof: One round protocol for 3SAT, where a prover returns a satisfying assignment for the input if it exists.





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- Lemma: **dIP = NP**

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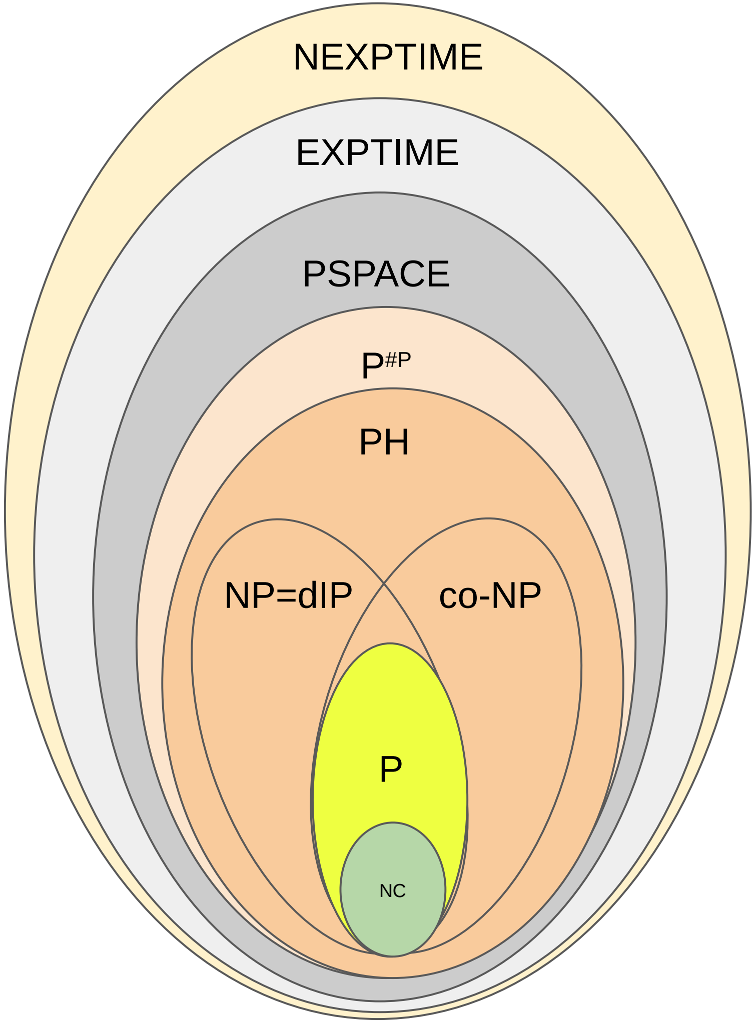

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IP: Probabilistic Verifier

Definition [GMR '89]: Probabilistic Verifiers and **IP**

For $k \geq 1$, we say that a language L has in **IPTIME**[k] if there's a probabilistic poly-time TM V that has a k-round interaction with P: $\{0,1\}^* \to \{0,1\}^*$ that on input x

$$x \in L \Rightarrow \exists P \; \Pr_r[\mathrm{out}_V \langle V, P\rangle(x) = 1] \geq 2/3 \text{ (Completeness)}$$

$$x \notin L \Rightarrow \forall P \; \Pr_r[\mathrm{out}_V \langle V, P\rangle(x) = 1] \leq 1/3 \text{ (Soundness)}$$

The probabilities over the random bits r of V.

The class IP is defined as **IP** = $\bigcup_{c>0}$ **IPTIME**$[n^c]$

P:BPP::NP:IP

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  Additionally, we can also do all repetitions in parallel by asking multiple questions in each round, thereby decreasing the number of rounds.

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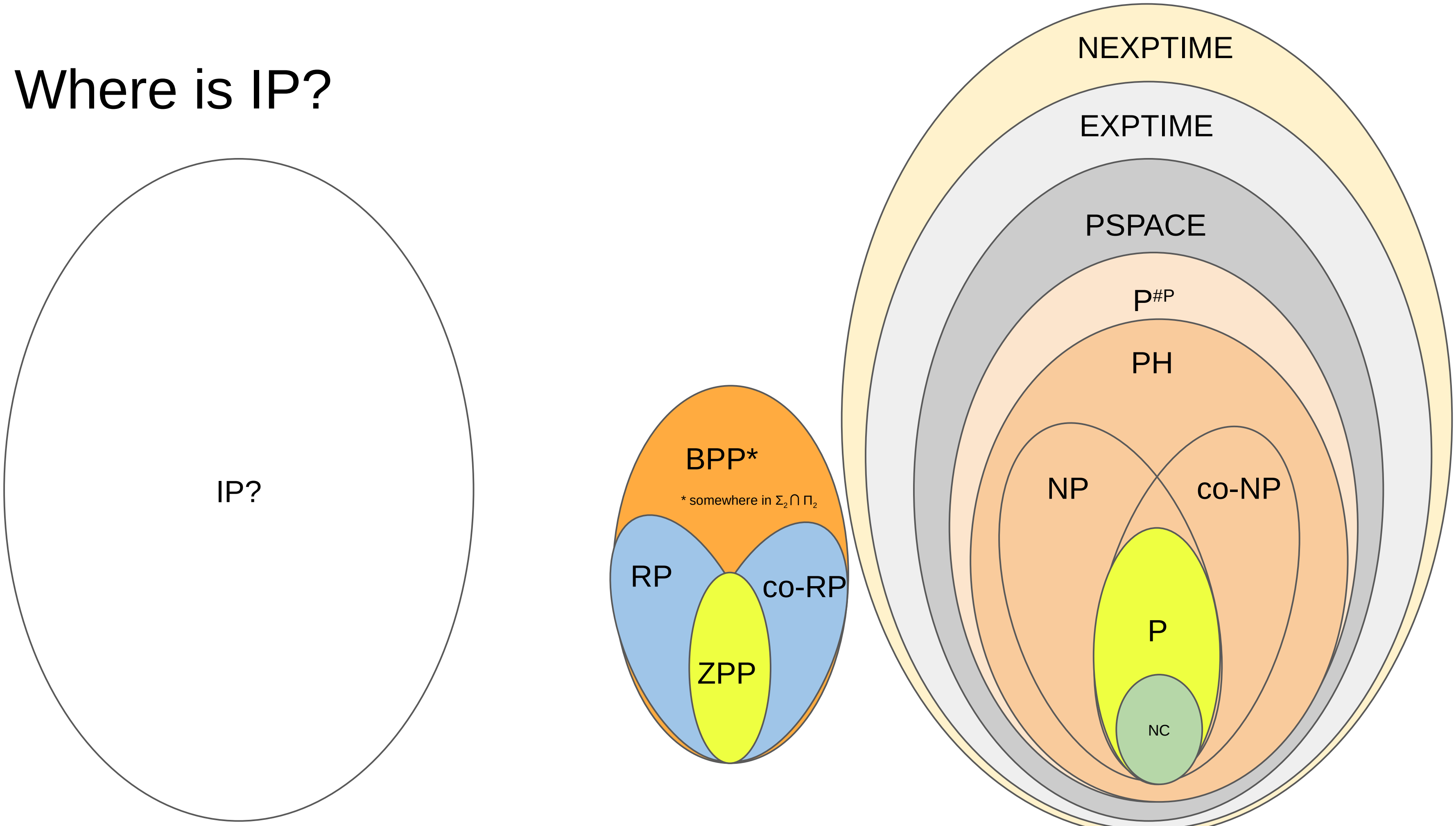

IP?
BPP*
* somewhere in Σ2 ∩ Π2
RP
co-RP
ZPP
NEXPTIME
EXPTIME
PSPACE
P#P
PH
NP
co-NP
P
NC

What's in IP?

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As dIP is in IP

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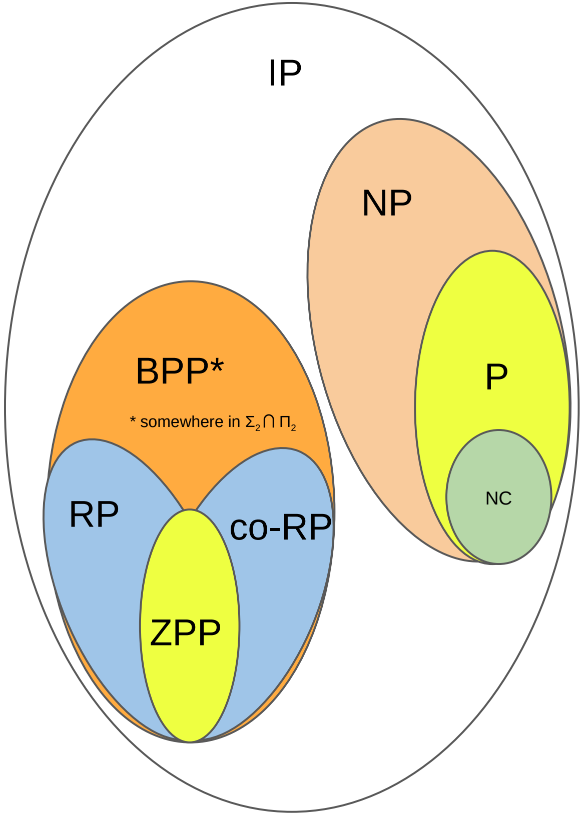

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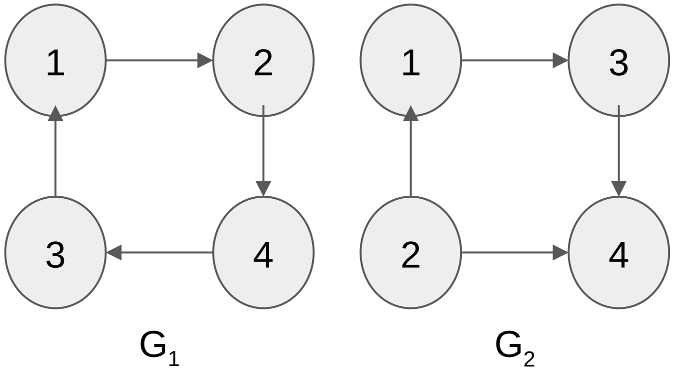
- Lemma: NONISO $\in$ IP [GMW '91]

NONISO in IP: Private Coin Protocol

Private Coin Protocol

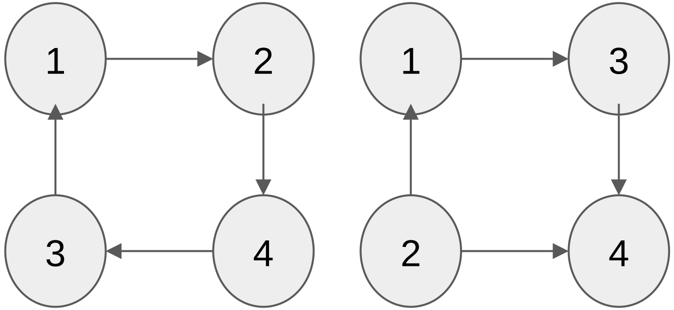
1. V randomly picks a graph between G_1 and G_2, say G_i. Randomly permute vertices of G_i to make H. Send H to P and asks if H is isomorphic to G_1 or G_2
2. Prover tries to figure out whether H is isomorphic to G_1 or G_2, sends $j \in \{1,2\}$ to V
3. V accepts if j==i.

NONISO in IP: Private Coin Protocol



NONISO in IP: Private Coin Protocol

V



G_1 G_2

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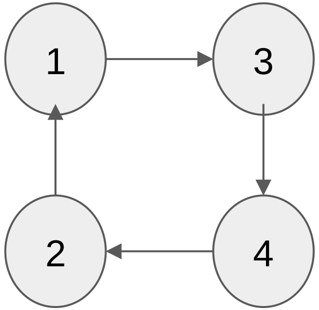
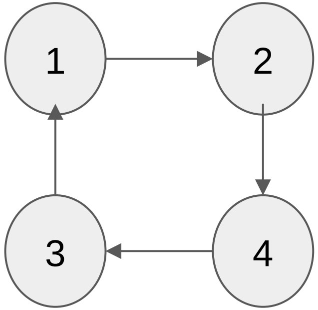


H

V



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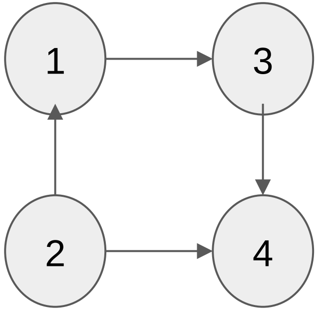

G_2

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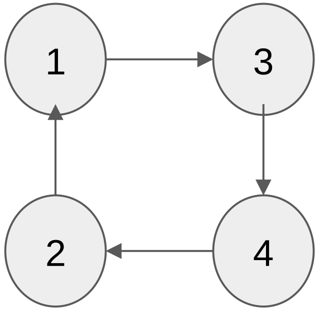

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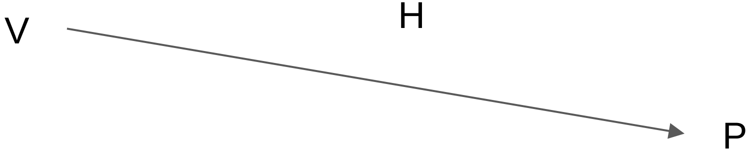

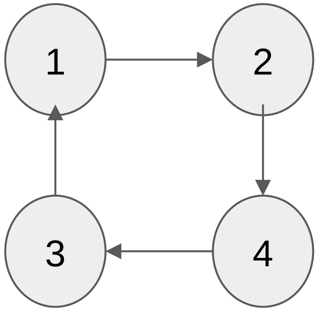

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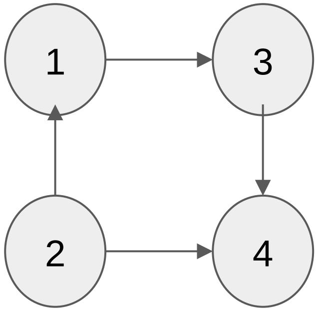

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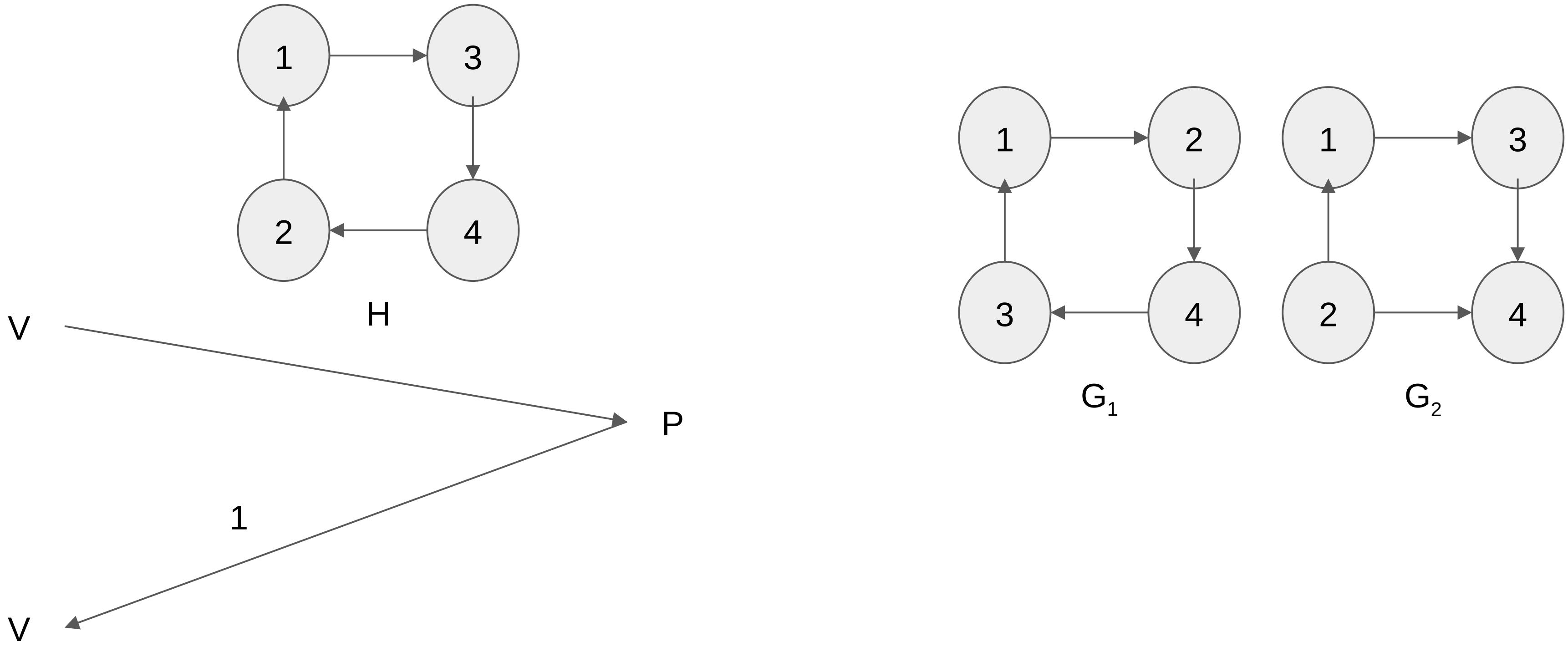

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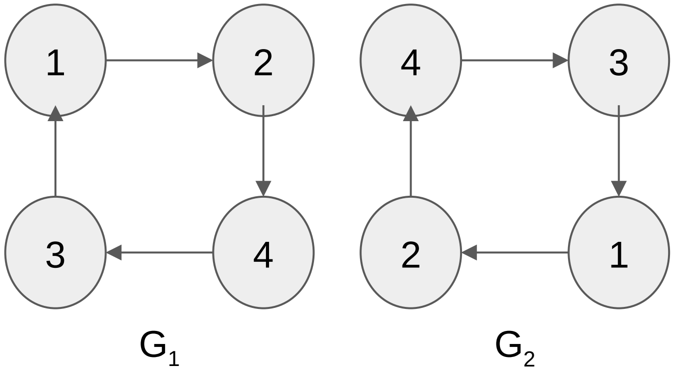

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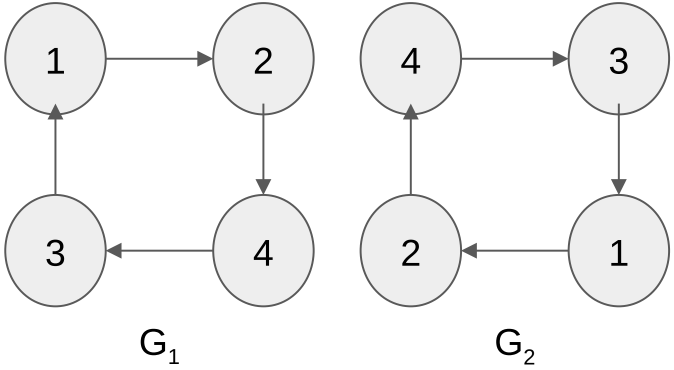

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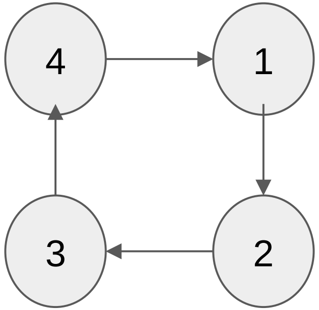

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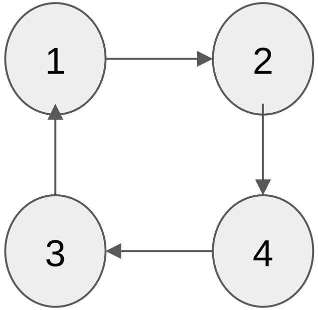

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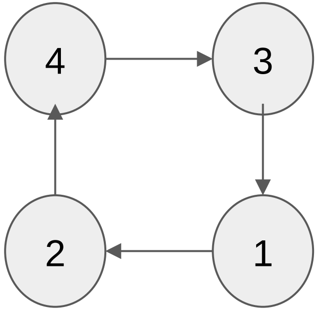

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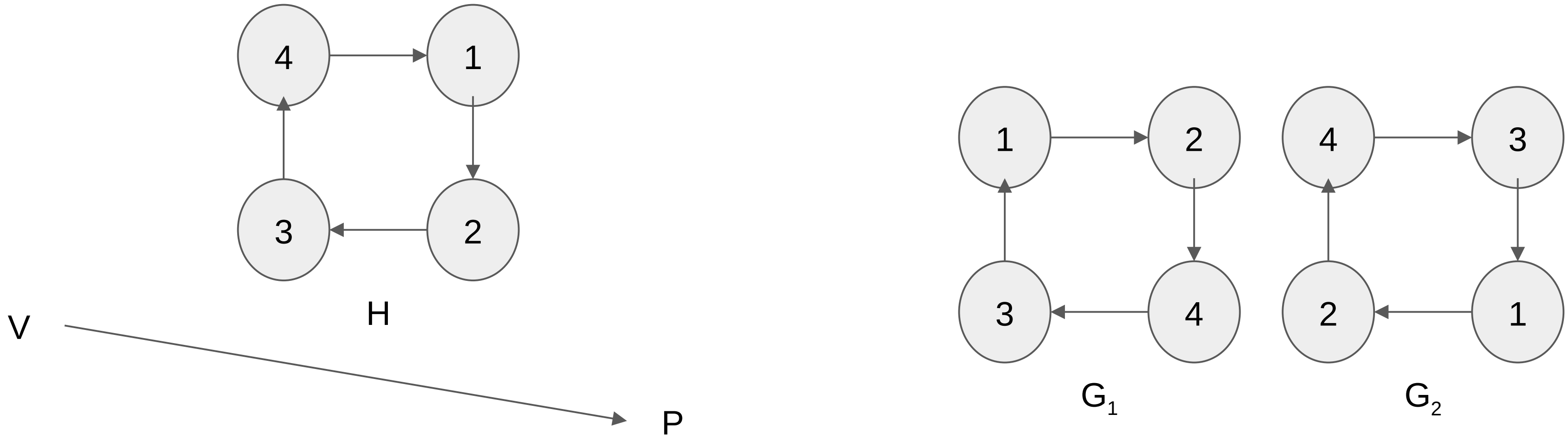

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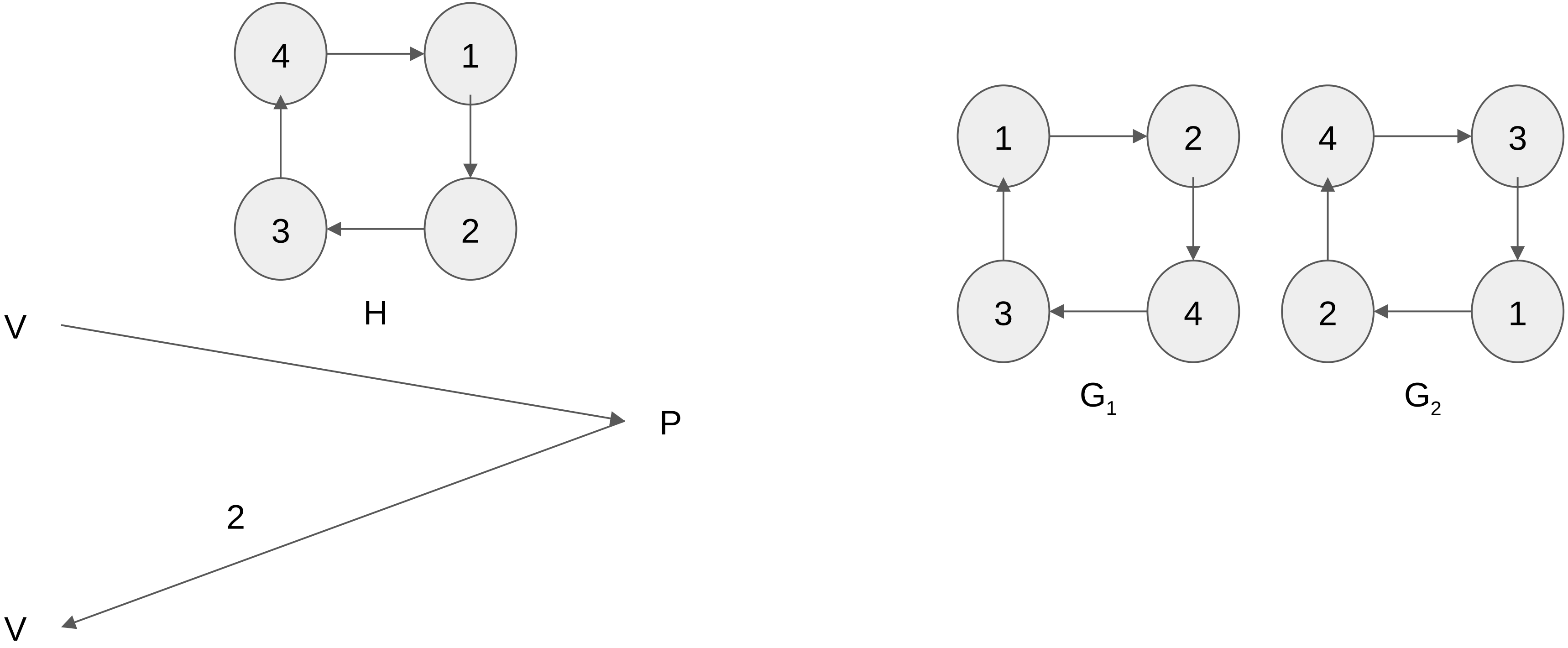

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- The probability of acceptance when the string is in the language is 1. (Perfect Completeness)

- If they are not isomorphic, the best the prover can do is to guess at random. So the probability of acceptance when it isn't in the language is ½. We can decrease this be multiple repetitions.

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As dIP is in IP

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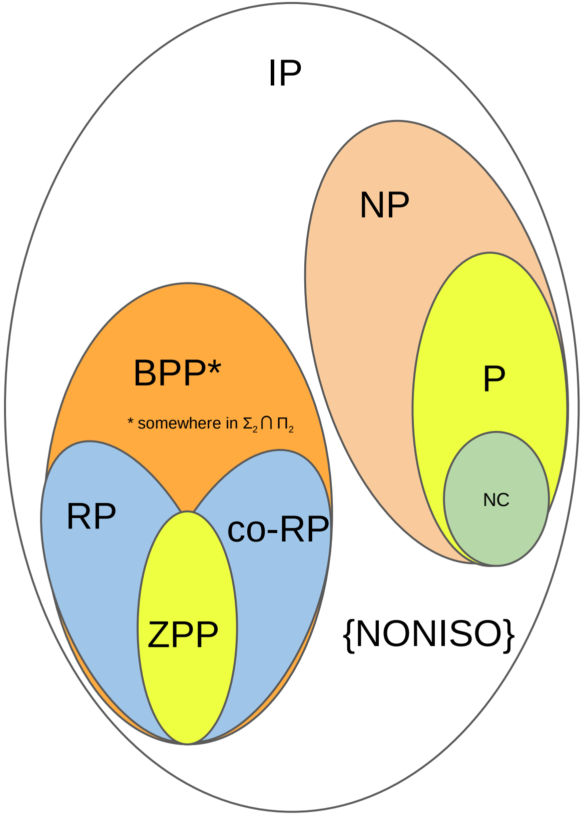

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Obs 4: The transcript must be "short"

Obs 5: Both V and P have access to the input x

Interactive Proof systems

Obs 1: If the prover(P) is g and the verifier(V) f

Who starts?

Probabilistic?

Obs 2: The prover <u>must</u> be all powerful

Obs 3: The verifier should be "efficient"

Probabilistic poly-time.

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What about random bits of V?

AM and MA: Public and Private coins

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Constant round interactive proofs with public coins: **AM** and **MA.**

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Verifier starts

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- Theorem: **BP.NP** = **AM** $\subseteq \boldsymbol{\Sigma_3}$

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- Theorem [Babai ‘88]: **AM[k]** = **AM[2]** for constant k

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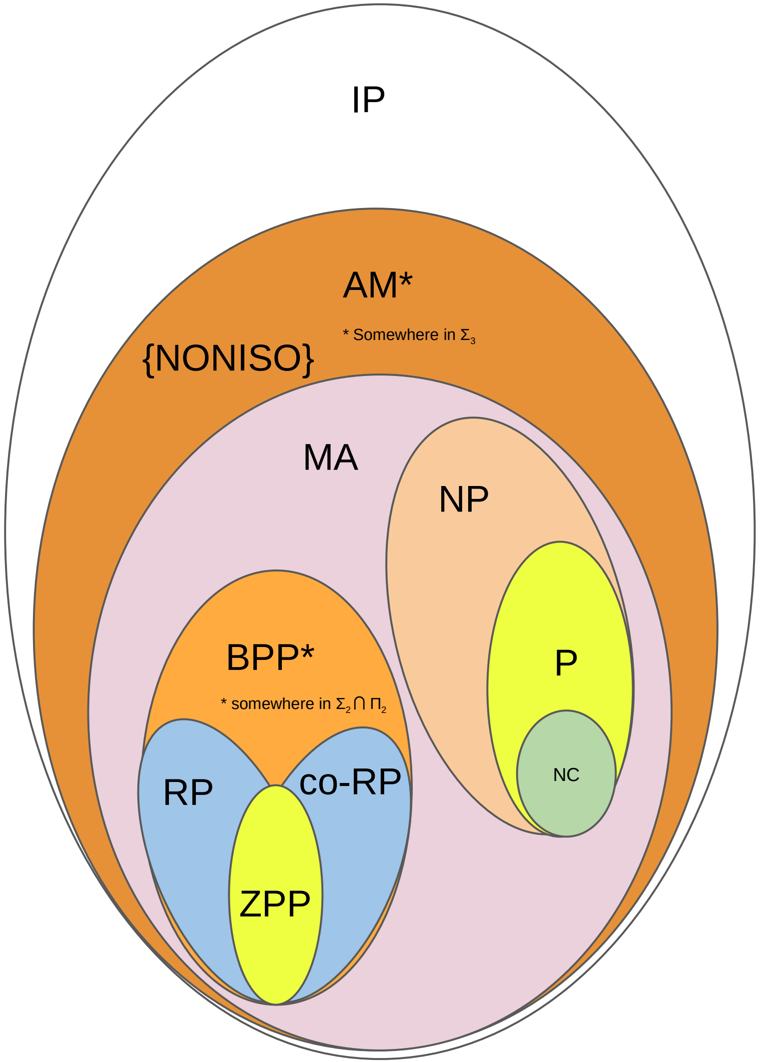
- Theorem: **BP.NP** = **AM** $\subseteq \boldsymbol{\Sigma_3}$
- Theorem [Babai '88]: **AM[k]** = **AM[2]** for constant k
- Theorem [GS '86]: **AM[k]** $\subseteq$ **IP[k]** $\subseteq$ **AM[k+2]** for polynomial k.

What’s in IP?


IP
AM*
* Somewhere in Σ_3
{NONISO}
MA
NP
BPP*
* somewhere in $\Sigma_2 \cap \Pi_2$
P
NC
RP
co-RP
ZPP

Theorem: IP $\subseteq$ PSPACE

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Proof Idea: Since we restrict certificates to be poly-size, it's easy to see that one can use a PSPACE machine to run through all possible transcripts to simulate a prover and calculate exactly the acceptance probability.

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Proof: Consider a language A in IP with a verifier V. Let the transcript be exactly of size $p = \text{poly}(n)$ for all inputs x of size n. We will construct a PSPACE machine M which decides A.

Theorem: IP $\subseteq$ PSPACE

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Definition: For any string x, we define

$$\Pr[\text{ V accepts x }] = \max_P \Pr[\text{ <V,P> accepts x }]$$

If x is in A, then it is at least ⅔ and at most ⅓ if it is not.

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Definition: $M_j = m_1,\ldots,m_j$ is the partial transcript upto length j. m_i represents the i^{th} message.

Definition: $\langle V,P\rangle(x,r,M_j)$ = accept, for a random string r of length p, if there exists $m_{j+1},..m_p$ such that

1. For $j \le i < p$ and i is even $V(x,r,M_i) = m_{i+1}$
2. For $j \le i < p$ and i is odd $P(x,M_i) = m_{i+1}$
3. m_p is accept

Theorem: IP $\subseteq$ PSPACE

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Obs: Using previous definitions,

$\Pr[\ \langle V,P\rangle$ accepts x starting at $M_j\ ] = \Pr[\ \langle V,P\rangle(x,r,M_j) = \text{accept}\ ]$ (1)

$\Pr[\ V$ accepts x starting at $M_j\ ] = \max_P \Pr[\ \langle V,P\rangle$ accepts x starting at $M_j\ ]$ (2)

Theorem: IP $\subseteq$ PSPACE

Obs: Using previous definitions,

Pr[<V,P> accepts x starting at M_j] = Pr[<V,P>(x,r,M_j) = accept] (1)

Pr[V accepts x starting at M_j] = $\max_P$ Pr[<V,P> accepts x starting at M_j] (2)

The goal is now to compute the probability of V accepting x starting from M_0. If this is greater than ⅔ then x must be in A, if it less than ⅓ then it must not be in A. We do this recursively.

Theorem: IP $\subseteq$ PSPACE

$N_{M\_j} = 0$ if $j = p$ and m_p = reject

$= 1$ if $j = p$ and m_p = accept

$= \max_{m\_\{j+1\}} N_{M\_\{j+1\}}$ odd $j < p$

$=$ wt-avg$_{m_\{j+1\}}$ $N_{M\_\{j+1\}}$ even $j<p$

wt-avg$_{m_\{j+1\}}$ $N_{M\_\{j+1\}} = \sum_{m\_\{j+1\}} ((\Pr[V(w,r,M_j)=m_{j+1}]) \cdot N_{M\_\{j+1\}})$

Theorem: IP $\subseteq$ PSPACE

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Claim 1: N_{M_j} = Pr[V accepts x starting at M_j]

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We need to prove the following 2 claims, with that the proof is complete.

Theorem: IP $\subseteq$ PSPACE

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Base case: j = p. The last message must be *accept* or *reject.* Hence, the probability of acceptance when the last message is *reject* is 0 and when the last message is *accept*, it is 1. This is exactly how N_{M_j} is defined.

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Inductive step: Assume the claim to be true for some $j+1 \leq p$. We have 2 cases, one when j is even and when j is odd.

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IH: $N_{M\_\{j+1\}}$ = Pr[V accepts x starting at M_{j+1}]

Theorem: IP $\subseteq$ PSPACE

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Theorem: IP $\subseteq$ PSPACE

When j is odd, the message m_{j+1} is from P to V. From the definition of N_{M_j}

$N_{M\_j} = \max_{m\_\{j+1\}} N_{M\_\{j+1\}}$

$N_{M\_j} = \max_{m\_\{j+1\}}$ Pr[V accepts x starting at M_{j+1}]

$= \max_{m\_\{j+1\}} \max_{P'}$ Pr[<V,P'>(x,r,M_{j+1}) = accept]

1… $\leq \max_P$ Pr[<V,P> accepts x starting at M_j], P can send the maximizing m^*_{j+1}

2… $\geq \max_P$ Pr[<V,P> accepts x starting at M_j], P cannot be better than P'

Therefore,

N_{M_j} = Pr[V accepts x starting at M_j]

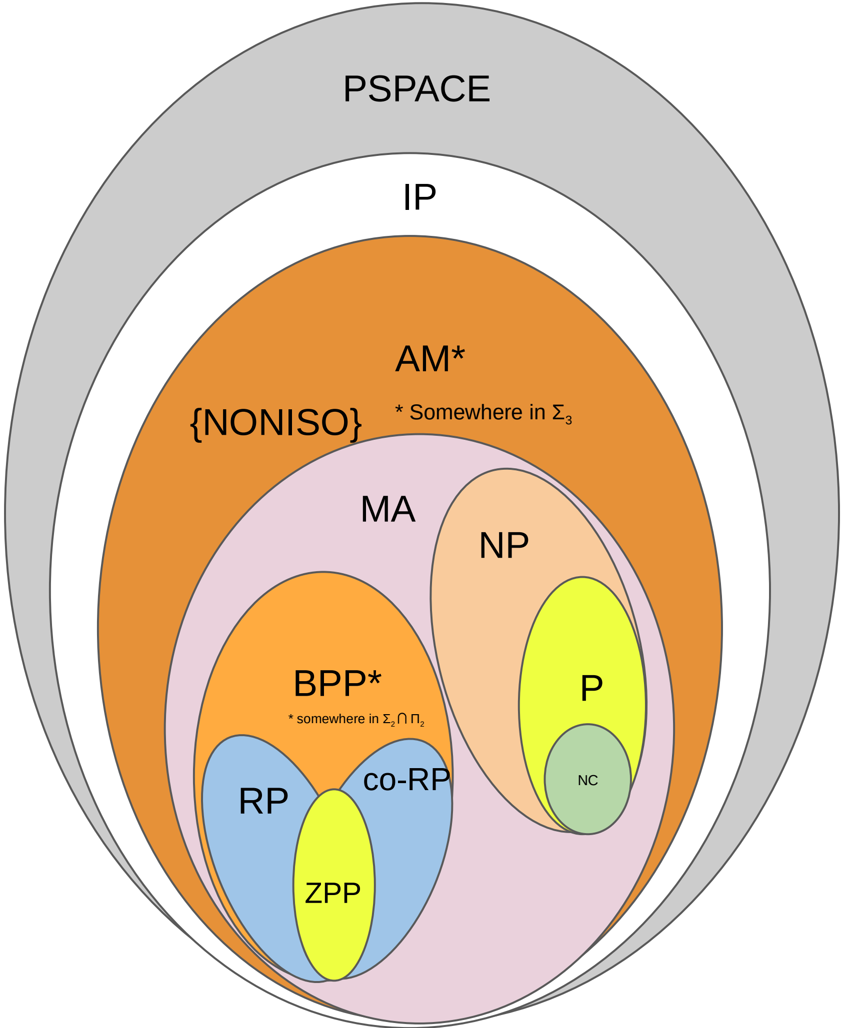
Theorem: IP $\subseteq$ PSPACE

Claim 2: N_{M_j} can be calculated in PSPACE

From the above proof, it also clear that these values can be calculated in PSPACE recursively. The depth of the recursion would be p. M calculates N_{M_j} for every j and M_j.

$\square$

Where is IP?



PSPACE
IP
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- Proven by [LFKN '92]

#3SAT Prerequisites

- Definition: #3SAT

#3SAT = {(ϕ,k)| where ϕ is a 3CNF with exactly k satisfying assignments}

#ϕ is the number of satisfying assignments of 3CNF ϕ

Say $\phi(x_1,...x_n)$, then

$$\#\phi = \Sigma_{b\_1 \in \{0,1\}}\Sigma_{b\_2 \in \{0,1\}}...\Sigma_{b\_n \in \{0,1\}}\ \phi(b_1,...b_n)$$

$\phi(b_1,...b_n) = 1$ if $b_1 \ldots b_n$ is a satisfying assignment, 0 otherwise

We define $\#\phi(a_1,...a_{i-1})$ as

$\#\phi(a_1,...a_{i-1}) = \Sigma_{b\_i \in \{0,1\}}...\Sigma_{b\_n \in \{0,1\}}\phi(a_1,...,a_{i-1},b_i,...,b_n)$

#3SAT Prerequisites

Observation*: $\#\phi(a_1,...a_{i-1}) = \#\phi(a_1,...a_{i-1},0) + \#\phi(a_1,...a_{i-1},1)$

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$= \Sigma_{b\_\{i+1\} \in \{0,1\}}...\Sigma_{b\_n \in \{0,1\}}\phi(a_1,...,a_{i-1},0,...,b_n) + \Sigma_{b\_\{i+1\} \in \{0,1\}}...\Sigma_{b\_n \in \{0,1\}}\phi(a_1,...,a_{i-1},1,...,b_n)$

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3. Step 2: Verifier sets x_1 to 0 in ϕ (ϕ_1) and x_1 to 1(ϕ_2) and evaluates ϕ_1 and ϕ_2 and asks the verifier for $\#\phi_1$ and $\#\phi_2$

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3. Step 2: Verifier sets x_1 to 0 in ϕ (ϕ_1) and x_1 to 1(ϕ_2) and evaluates ϕ_1 and ϕ_2 and asks the verifier for $\#\phi_1$ and $\#\phi_2$
4. Step 3: Prover sends k_1 and k_2
5. Step 4: Verifier verifies that $K = k_1 + k_2$
6. Repeat by setting each variable x_i to 0 and 1 and verifying
7. Step ??: Once all variables have been set, Verifier asks the prover the number of satisfying assignments and also verifies the answer by itself.

#3SAT $\subseteq$ IP? [Attempt 1]

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$\phi(x_1,..x_n)$

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$\phi(x_1,..x_n)$ K

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$\phi(x_1,..x_n)$ K

$\phi(x_1,..x_n) \longrightarrow \phi(0,..x_n)$

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$\phi(0,..x_n)$ k1

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$\phi(1,..x_n)$

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$\phi(0,..x_n)$ k1

$\phi(1,..x_n)$ k2

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$\phi(x_1,..x_n)$ K

$\phi(0,..x_n)$ k1

$\phi(1,..x_n)$ k2

K=k1+k2?

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$\phi(0,..x_n)$ k1

K=k1+k2?

$\phi(1,..x_n)$ k2

$\phi(0,0,..x_n)$

$\phi(0,1,..x_n)$

#3SAT $\subseteq$ IP? [Attempt 1]

$\phi(x_1,..x_n)$ K

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$\phi(1,..x_n)$ k2

K=k1+k2?

$\phi(0,0,..x_n)$ k4

$\phi(0,1,..x_n)$ k5

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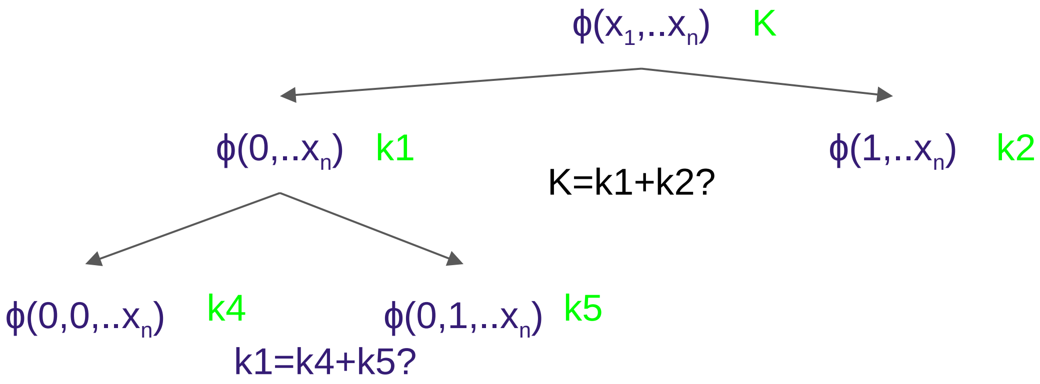

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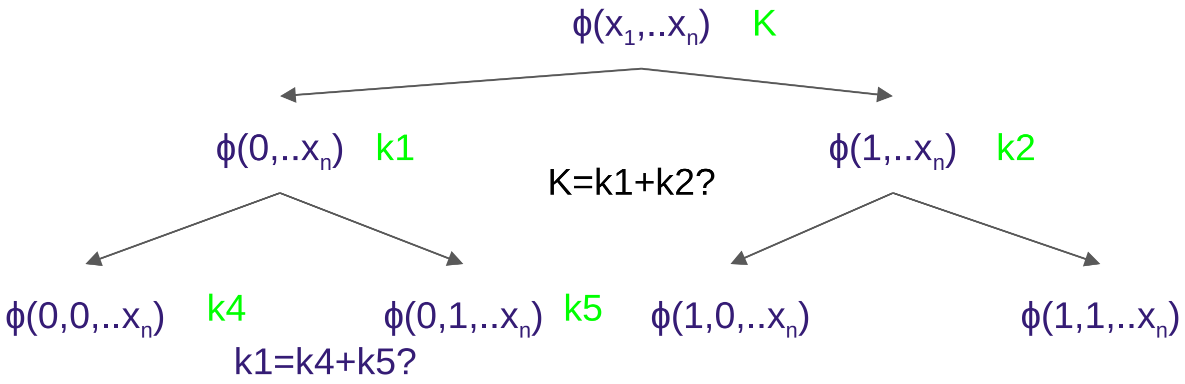

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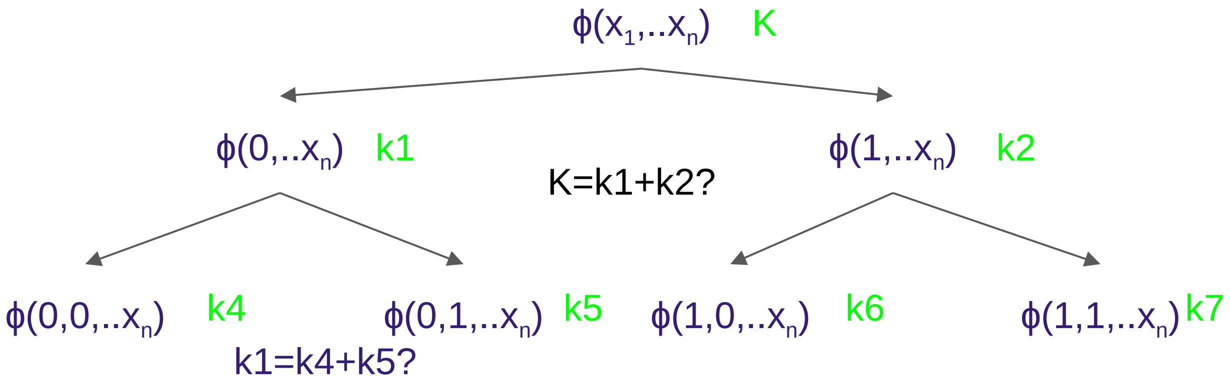

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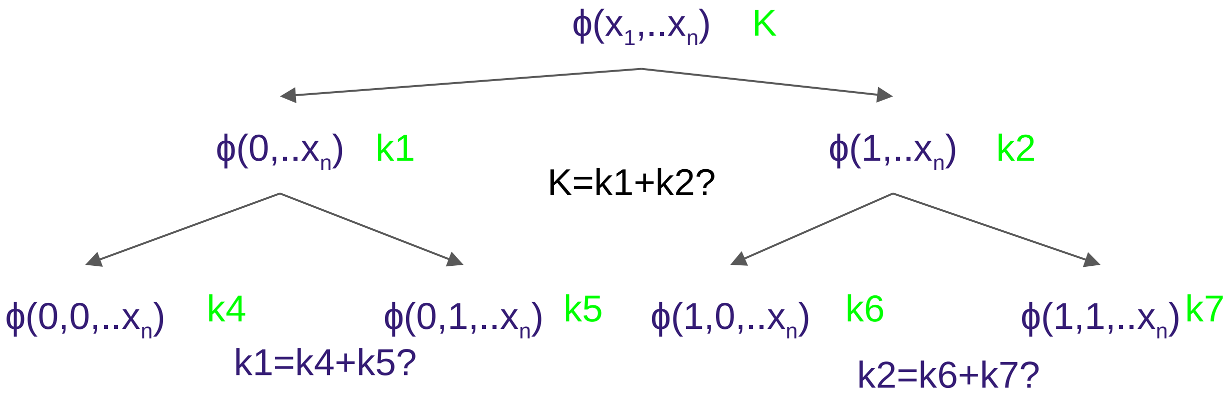

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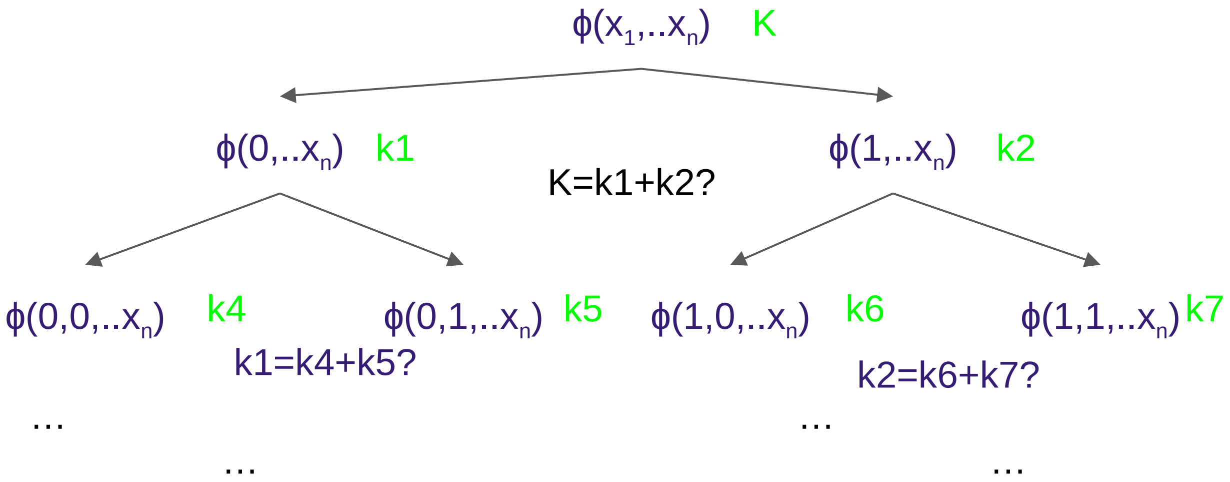

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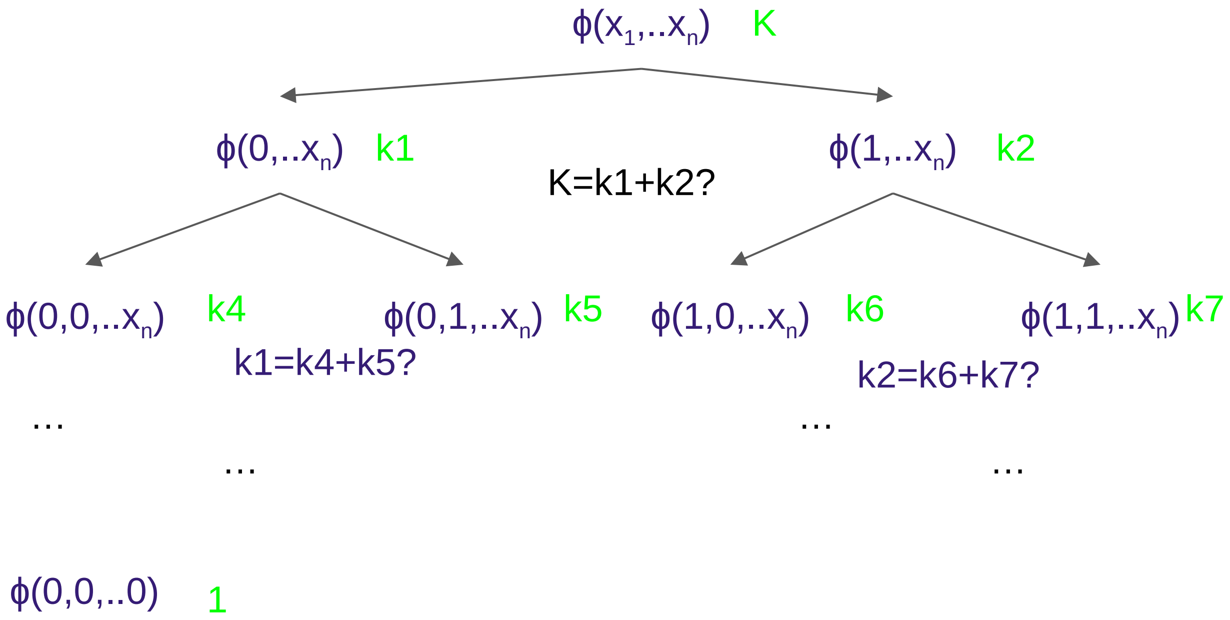

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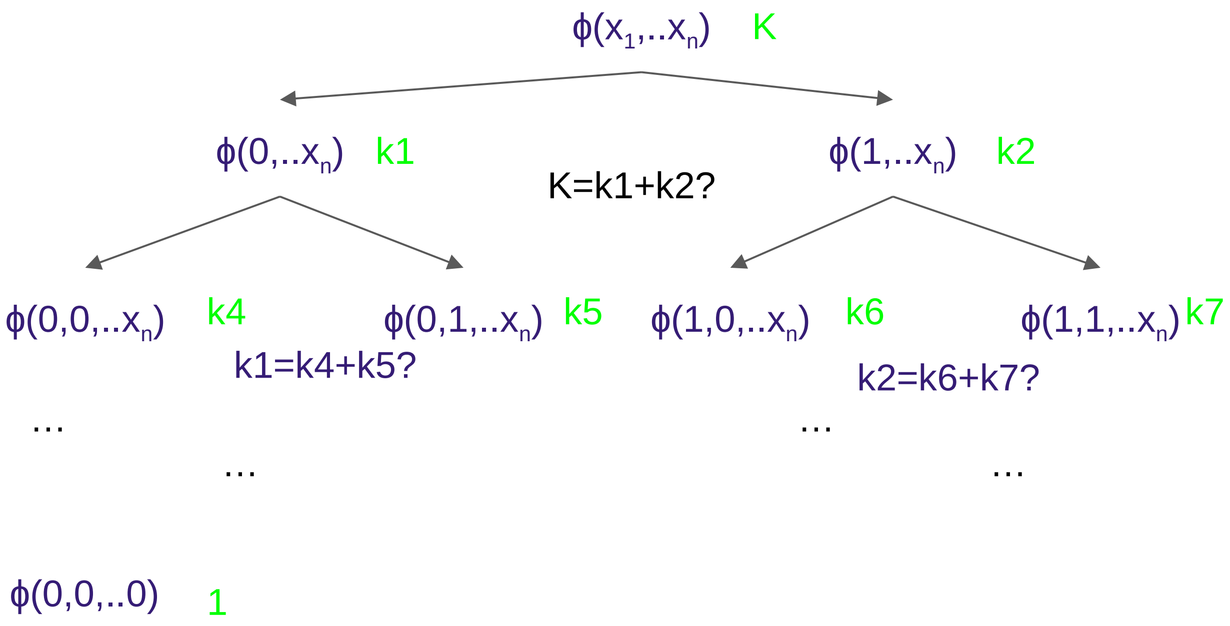

$\phi(0,0,..0)$=1 == 1

#3SAT $\subseteq$ IP? [Attempt 1]

$\phi(x_1,..x_n)$ K

$\phi(0,..x_n)$ k1

K=k1+k2?

$\phi(1,..x_n)$ k2

$\phi(0,0,..x_n)$ k4

$\phi(0,1,..x_n)$ k5

$\phi(1,0,..x_n)$ k6

$\phi(1,1,..x_n)$ k7

k1=k4+k5?

k2=k6+k7?

…

…

…

…

$\phi(0,0,..0)$ 1

$\phi(0,1,..1)$ 0

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$\phi(1,..x_n)$ k2

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$\phi(0,0,..x_n)$ k4

$\phi(0,1,..x_n)$ k5

$\phi(1,0,..x_n)$ k6

$\phi(1,1,..x_n)$ k7

k1=k4+k5?

k2=k6+k7?

…

…

…

…

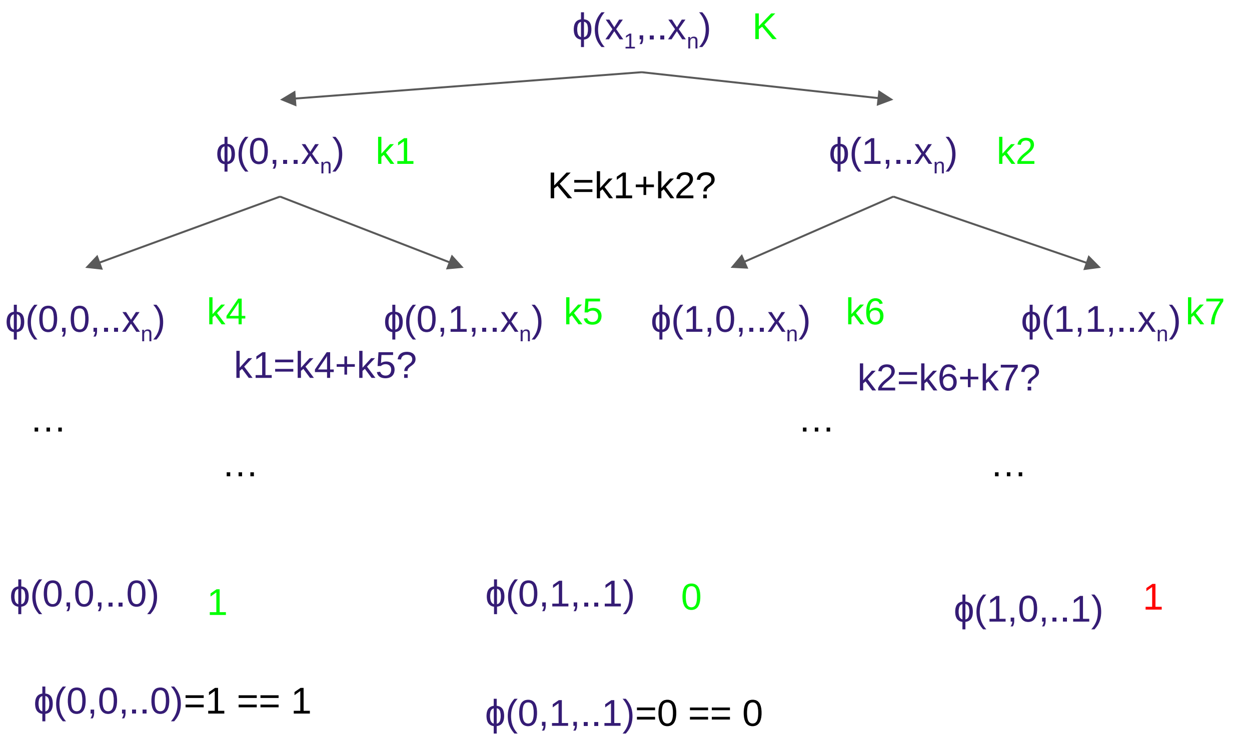
$\phi(0,0,..0)$ 1

$\phi(0,1,..1)$ 0

$\phi(0,0,..0)$=1 == 1

$\phi(0,1,..1)$=0 == 0

#3SAT ⊆ IP? [Attempt 1]



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$\phi(x_1,..x_n)$ K

$\phi(0,..x_n)$ k1

K=k1+k2?

$\phi(1,..x_n)$ k2

$\phi(0,0,..x_n)$ k4

$\phi(0,1,..x_n)$ k5

k1=k4+k5?

$\phi(1,0,..x_n)$ k6

$\phi(1,1,..x_n)$ k7

k2=k6+k7?

…

…

…

…

$\phi(0,0,..0)$ 1

$\phi(0,1,..1)$ 0

$\phi(1,0,..1)$ 1

$\phi(0,0,..0)$=1 == 1

$\phi(0,1,..1)$=0 == 0

$\phi(1,0,..1)$=0 != 1

#3SAT $\subseteq$ IP? [Attempt 1]

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Idea: Randomly choose a path in the tree

#3SAT $\subseteq$ IP? [Attempt 2]

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$\phi(0,..x_n)$ k1

#3SAT $\subseteq$ IP? [Attempt 2]

$$\begin{tikzcd}[column sep=small] & \phi(x_1,..x_n) \quad K \arrow[dl] \arrow[dr] & \\ \phi(0,..x_n) \quad k1 & & \phi(1,..x_n) \end{tikzcd}$$

#3SAT $\subseteq$ IP? [Attempt 2]

$$\begin{array}{ccc} & \phi(x_1,..x_n) \quad \color{lime}{K} & \\ \swarrow & & \searrow \\ \phi(0,..x_n) \quad \color{lime}{k1} & & \phi(1,..x_n) \quad \color{lime}{k2} \end{array}$$

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#3SAT $\subseteq$ IP? [Attempt 2]

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k2=k6+k7?

…

$\phi(1,0,..1)$ 1

$\phi(1,0,..1)=0$!= 1

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- The probability that the prover actually gets caught is 2^{-n}. We need to catch every wrong branch at every step.

- So, we always accept when the number of satisfying assignments are correct, but we will also accept when it is incorrect with probability $1 - 2^{-n}$.

Boolean is F_2

Boolean is F_2

Every boolean formula can be expressed as a polynomial over elements of F_2

We use the following trick:

$$a \land b \equiv ab$$

$$a \lor b \equiv 1 - (1-a)(1-b) \equiv a + b - ab$$

$$\neg a \equiv (1-a)$$

$$\text{True} \equiv 1$$

$$\text{False} \equiv 0$$

Example:

$$(x_1 \lor x_3 \lor \neg x_4) \equiv (x_1 + x_3 - x_1 x_3) + (1-x_4) - (x_1 + x_3 - x_1 x_3)(1-x_4)$$

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- We are now able to express a boolean formula ϕ as a polynomial P_ϕ

- The degree of each clause will be at most 3, as ϕ is a 3CNF, and the net degree will be at most 3m where there are m clauses in ϕ.

- The size of the polynomial will also be bound polynomial in the size of ϕ as we don't need to expand the terms

Boolean is F_2

We can restate our equations as follows, where X_is are now formal variables

$$\#\phi = \Sigma_{b\_1 \in \{0,1\}} \Sigma_{b\_2 \in \{0,1\}} \ldots \Sigma_{b\_n \in \{0,1\}} P_\phi(b_1,\ldots b_n)$$

$$\#\phi(X_1,\ldots X_{i-1}) = \Sigma_{b\_i \in \{0,1\}} \ldots \Sigma_{b\_n \in \{0,1\}} P_\phi(X_1,\ldots,X_{i-1},b_i,\ldots,b_n)$$

$$\#\phi(X_1,\ldots X_i) = \#\phi(X_1,\ldots X_{i-1},0) + \#\phi(X_1,\ldots X_{i-1},1)$$

Theorem[LFKN ‘92]: #3SAT $\in$ IP

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Key Idea: Arithmetization

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Theorem[LFKN '92]: #3SAT $\in$ IP

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Once we do that, we can plug in any element in F_p into our polynomial

Theorem[LFKN ‘92]: #3SAT $\in$ IP

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- How large should p be?

The number of satisfying assignments can be at most 2^n, therefore, we can chose a prime between 2^n and 2^{2n}.

We ask the prover to provide this prime at the start of the protocol and the verifier can verify primalty in polynomial time.

Sumcheck protocol

A generic protocol to verify equations of the form

$$K = \Sigma_{b\_1 \in \{0,1\}} \ldots \Sigma_{b\_n \in \{0,1\}}\, g(X_1,\ldots X_n) \qquad \ldots \text{ eq(1)}$$

Where g is any polynomial of small size and which can be evaluated in polynomial time.

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Obs: P_ϕ is a degree 3m polynomial it’s size is of the order of the size of ϕ. It can also be easily evaluated in the same way we evaluate formulas on assignments. So we can use the sumcheck protocol.

Sumcheck protocol

Sumcheck protocol

Obs: $h(X_1) = \Sigma_{b\_2 \in \{0,1\}} ... \Sigma_{b\_n \in \{0,1\}}\, g(X_1, b_2, ... b_n)$

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Is a univariate polynomial of degree at most m in the variable X_1.

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The input to the protocol would be a polynomial $g(X_1,\ldots,X_n)$ and K.

Obs: g can be evaluated in polynomial time, however h cannot even be computed in polynomial time

Sumcheck protocol

Input: $g(X_1,...X_n)$, K

V: if n = 1, verify K = g(0) + g(1)

V: It asks the prover to send a polynomial h, as defined previously, a polynomial in X_1

P: sends a polynomial s

V: verify that s(0) + s(1) = K. Selects a random element from F_p, say a. It calculates s(a).

Recursively solve with the input as

$g(a,X_2,...X_n)$ and s(a).

Sumcheck protocol

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Sumcheck protocol

$g(X_1,..X_n)$

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...

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Analysis of protocol

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- Sending univariate polynomials is sending d numbers where d is the degree of the polynomial.

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- How lucky does the prover need to be for the verifier to accept an incorrect string?

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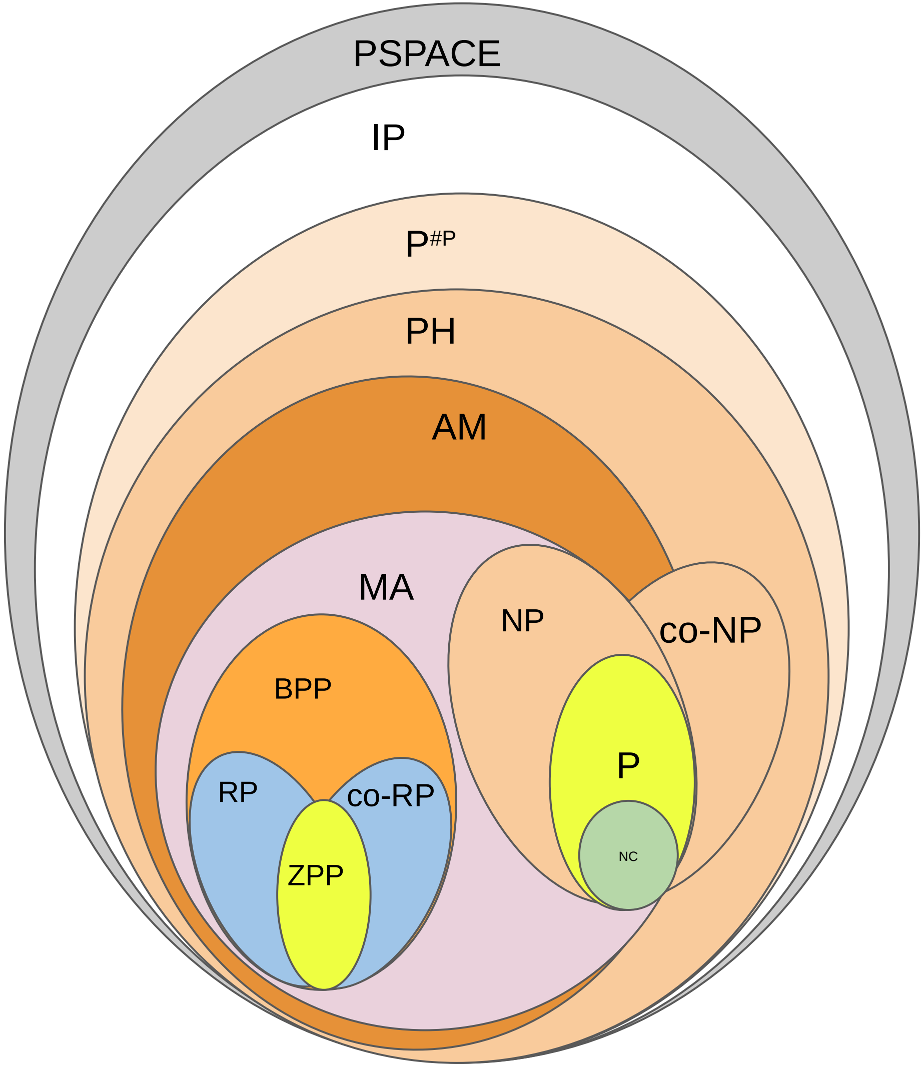
Therefore the error probability is less than $3n^2/2^n$ which is less than ⅓ for n>9

What's in IP?



TQBF $\subseteq$ IP?

Definition: TQBF

$$\text{TQBF} = \{\ \Psi = Q_1x_1 \ldots Q_nx_n\ \phi(x_1,\ldots x_n) \mid \Psi = \text{True},\ Q_i \text{ in } \{\exists,\forall\}, \text{ boolean formula } \phi\}$$

$\Psi = \forall x_1,\exists x_2,\forall x_3 \ldots \exists x_n\ \phi(x_1,\ldots x_n) \in \text{TQBF}$ iff

$$\Pi_{b\_1 \in \{0,1\}} \Sigma_{b\_2 \in \{0,1\}} \Pi_{b\_3 \in \{0,1\}} \ldots \Sigma_{b\_n \in \{0,1\}} P_\phi(b_1,\ldots b_n) = 1$$

Where P_ϕ is the polynomial as defined before over F_2

Sumcheck Protocol?

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Obs 2: Multiply over ∀

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As for 3SAT, when we need a univariate polynomial over a variable quantified by ∃, we must check the additivity, i.e, $s(0)+s(1) = K$

Obs 2: Multiply over ∀

When we have a univariate polynomial over a variable quantified by ∀, we must check multiplicity, i.e, $s(0) \cdot s(1) = K$

Sumcheck Protocol?

- Unlike adding polynomials, multiplying polynomials increase the degree

- If we define $h(X_1)$ as defined previously:

$$h(X_1) = \Sigma_{b\_2 \in \{0,1\}} \Pi_{b\_3=\{0,1\}} \ldots \Sigma_{b\_n \in \{0,1\}} P_{\phi}(X_1, \ldots b_n)$$

This can have degree at most 2^n. Which cannot be sent from the prover to the verifier.

Obs:

$$x^k = x \text{ in } F_2 \text{ for any } k > 0$$

Linearization

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Any polynomial $p(X_1,...X_n)$ can be converted to a *multilinear* polynomial $q(X_1,...X_n)$ where

1. The degree of any variable in any term of q is at most 1
2. $p(a_1,...a_n) = q(a_1,...a_n)$ for any $a_1 \ldots a_n \in \{0,1\}$

Linearization

Definition: Linearization operator L

$L_i(p) = X_i \cdot p(X_1,\ldots,X_{i-1},1,X_{i+1},\ldots X_n) + (1-X_i) \cdot p(X_1,\ldots,X_{i-1},0,X_{i+1},\ldots X_n)$

Defines a new polynomial such that

1. Degree of X_i in $L_i(p)$ is at most 1
2. $L_i(p)$ gives the same values as p for all binary inputs

Obs: $q = L_1(L_2(\ldots L_n(p)\ldots)))$

Linearization

Definition: ∀ operator for polynomials

$$\forall_i\, p(X_1,\ldots X_n) = p(X_1,\ldots,X_{i-1},0,X_{i+1},\ldots X_n) \cdot p(X_1,\ldots,X_{i-1},1,X_{i+1},\ldots X_n)$$

Definition: ∃ operator for polynomials

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Linearization

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Original polynomial:

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$$\Pi_{b\_1 \in \{0,1\}} \Sigma_{b\_2 \in \{0,1\}} \Pi_{b\_3 \in \{0,1\}} \ldots \Sigma_{b\_n \in \{0,1\}} P_\phi(b_1,\ldots b_n) = 1$$

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$$\forall_1\exists_2\forall_3\ldots\exists_n P_\phi(X_1,...X_n) = 1$$

Since we only care about using {0,1} to $P_\phi(X_1,...X_n)$, we do not lose semantics by adding linearization operators in between,

$$\forall_1 L_1 \ \exists_2 L_1 L_2 \ \forall_3\ldots\exists_n L_1 L_2\ldots L_n P_\phi(X_1,...X_n) = 1$$

The size of this expression is increased due to the addition of the linearization operator. The size will then be $O(n+1+2+...+n+|P_\phi|)$, which is still poly-size

Modified Sumcheck protocol

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Consider a polynomial $g(X_1,...X_n)$, we need to check whether

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Consider a polynomial $g(X_1,...X_n)$, we need to check whether

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Input: $R_1 R_2 \dots R_t g(X_1,...X_n)$ where R represents one of the 3 operators, t is poly(n) and a claim C.

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TQBF: g would be P_ϕ, t would be $o(n^3)$, and C would be 1

Modified Sumcheck protocol

V: provide a polynomial equal to $R_2 \ldots R_t g(X_1, \ldots X_n)$

P: returns a polynomial $s(X_1)$

V: 1) If $R_1 = \exists_1$ verify that $s(0) + s(1) = C$

2) If $R_1 = \forall_1$ verify that $s(0) \cdot s(1) = C$

3) If $R_1 = L_1$ and verify that $a \cdot s(1) + (1-a) \cdot s(0) = s(a)$

If all checks pass, pick a random element a, recursively prove that the polynomial $R_2 \ldots R_t g(a, \ldots X_n) = s(a)$

Modified Sumcheck protocol

$\forall_1 L_1\ \exists_2 L_1 L_2\ \forall_3 \ldots \exists_n L_1 L_2 \ldots L_n g(X_1, \ldots X_n)$

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$\forall_1 L_1 \exists_2 L_1 L_2 \forall_3 \ldots \exists_n L_1 L_2 \ldots L_n g(X_1,\ldots X_n)$ $\longrightarrow$ $s_1(X_1)$

Modified Sumcheck protocol

$\forall_1 L_1 \; \exists_2 L_1 L_2 \; \forall_3 \ldots \exists_n L_1 L_2 \ldots L_n g(X_1, \ldots X_n)$

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$s_1(0) \cdot s_1(1) == C$

Modified Sumcheck protocol

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$s_1(0) \cdot s_1(1) == C$ $\longleftarrow$

$L_1 \; \exists_2 L_1 L_2 \; \forall_3 \ldots \exists_n L_1 L_2 \ldots L_n g(a_1, X_2, \ldots X_n)$ $\longrightarrow$ $s_2(X_1)$

Modified Sumcheck protocol

$\forall_1 L_1\ \exists_2 L_1 L_2\ \forall_3 \ldots \exists_n L_1 L_2 \ldots L_n g(X_1,\ldots X_n)$

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$s_2(X_1)$

$(1 - a_1) \cdot s_2(0) + a_1 \cdot s_2(1) == s_2(a_1)$

Modified Sumcheck protocol

$$\forall_1 L_1\ \exists_2 L_1 L_2\ \forall_3 \ldots \exists_n L_1 L_2 \ldots L_n g(X_1,...X_n)$$

$$s_1(X_1)$$

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$s_3(X_2)$

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$(1 - a_{n-1}) \cdot s_t(0) + a_{n-1} \cdot s_t(1) == s_t(a_{n-1})$

Modified Sumcheck protocol

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…

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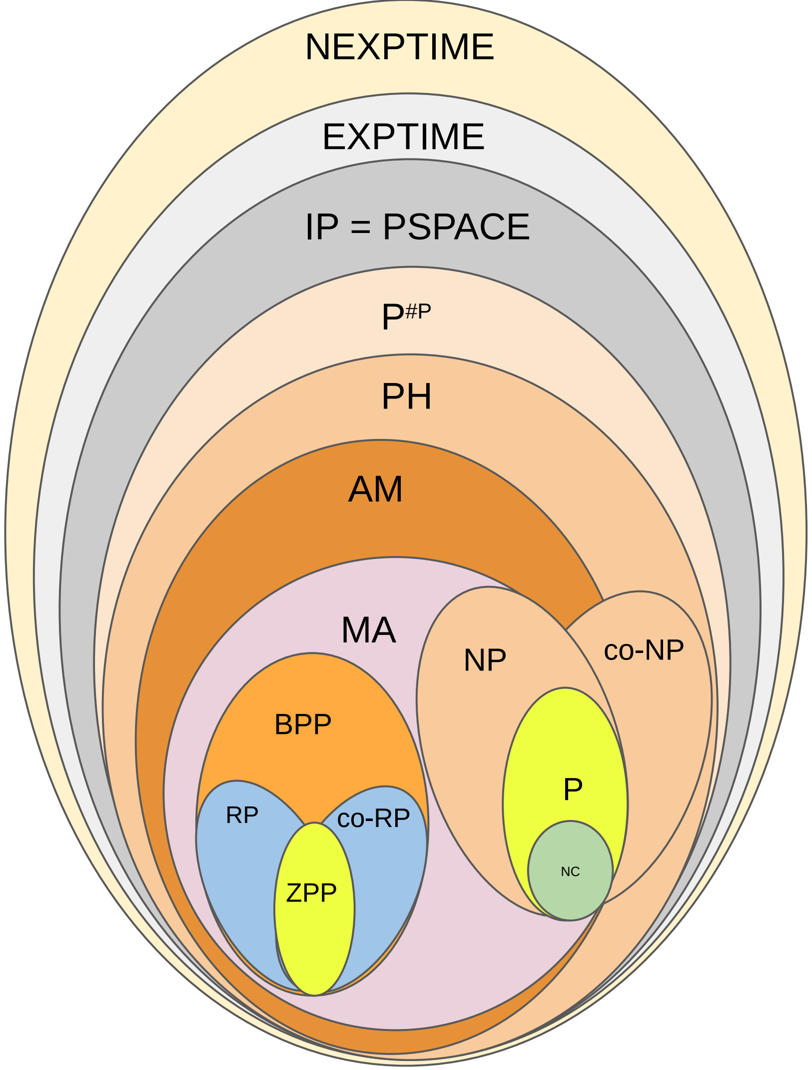
$g(a_1, a_2,\ldots a_n) == s_t(a_n)$

Where's IP?


NEXPTIME
EXPTIME
IP = PSPACE
P#P
PH
AM
MA
NP
co-NP
BPP
RP
co-RP
ZPP
P
NC

MIP

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- Theorem[BFL ‘91]: **MIP** = **MIP[2]** = **NEXPTIME**

QIP, MIP*

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- Theorem[JNVWY ‘20]: **MIP*** = **RE**

- We would be able to solve undecidable problems like the halting problem

IP = PSPACE Timeline

1985: AM, MA defined by Babai

1986: Goldwasser and Sipser show public coin private coin equivalence

1988: AM=AM[2] by BM, MIP is defined by BGKW

1989: IP is defined by GMR

1991: ZKP(NONISO in IP) by GMW, MIP=NEXP by BFL

1992: #3SAT in IP by LFKN, IP=PSPACE by Shamir, Simpler proof by Shen

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- Computational Complexity: A Modern Approach, by Sanjeev Arora and Boaz Barak. https://theory.cs.princeton.edu/complexity/book.pdf

TL; DR

- Randomness+Interaction is the key, alone they are “weak”
- Supreme power is useless unless succinct
- Mapping to polynomials is a very powerful technique